

# CRYSTALLINE COHOMOLOGY AND DE RHAM COHOMOLOGY

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**ABSTRACT.** The goal of this short paper is to give a slightly different perspective on the comparison between crystalline cohomology and de Rham cohomology. Most notably, we reprove Berthelot's comparison result without using pd-stratifications, linearisations, and pd-differential operators.

Crystalline cohomology is a  $p$ -adic cohomology theory for varieties in characteristic  $p$  created by Berthelot [Ber74]. It was designed to fill the gap at  $p$  left by the discovery [SGA73] of  $\ell$ -adic cohomology for  $\ell \neq p$ . The construction of crystalline cohomology relies on the *crystalline* site, which is a better behaved positive characteristic analogue of Grothendieck's *infinitesimal* site [Gro68]. The motivation for this definition comes from Grothendieck's theorem [Gro66] identifying infinitesimal cohomology of a complex algebraic variety with its singular cohomology (with  $\mathbb{C}$ -coefficients); in particular, infinitesimal cohomology gives a purely algebraic definition of the "true" cohomology groups for complex algebraic varieties. The fundamental structural result of Berthelot [Ber74, Theorem V.2.3.2] is a direct  $p$ -adic analogue of this reconstruction result: the crystalline cohomology of a smooth  $\mathbb{F}_p$ -variety  $X$  is identified with the de Rham cohomology of a lift of  $X$  to  $\mathbb{Z}_p$ , provided one exists. In particular, crystalline cohomology produces the "correct" Betti numbers, at least for liftable smooth projective varieties (and, in fact, even without liftability by [KM74]). We defer to [Ill94] for a detailed introduction, and connections with  $p$ -adic Hodge theory.

Our goal in this note is to give a different perspective on the relationship between de Rham and crystalline cohomology. In particular, we give a short proof of the aforementioned comparison result [Ber74, Theorem V.2.3.2]; see Theorem 3.6. Our approach replaces Berthelot's differential methods (involving stratifications and linearisations) with a resolutely Čech-theoretic approach. It seems that Theorem 3.2 is new, although it may have been known to experts in the field. This theorem also appears in forthcoming work of Alexander Beilinson [Bei].

**Conventions.** Throughout this note,  $p$  is a fixed prime number. Our base scheme will be typically be  $\Sigma = \text{Spec}(\mathbb{Z}_p)$ , though occasionally we discuss the theory over  $\Sigma_e = \text{Spec}(\mathbb{Z}_p/p^e)$  as well (for some  $e \geq 1$ ). All divided powers will be compatible with the divided powers on  $p\mathbb{Z}_p$ . Modules of differentials on divided power algebras are compatible with the divided power structure. A general reference for divided powers and the crystalline site is [Ber74].

## 1. REVIEW OF MODULES ON THE CRYSTALLINE SITE

Let  $S$  be a  $\Sigma$ -scheme such that  $p$  is locally nilpotent on  $S$ . The (small) crystalline site of  $S$  is denoted  $(S/\Sigma)_{\text{cris}}$ . Its objects are triples  $(U, T, \delta)$  where  $U \subset S$  is an open subset,  $U \subset T$  is a nilpotent thickening of  $\Sigma$ -schemes, and  $\delta$  is a divided power structure on the ideal of  $U$  in  $T$ ; the morphisms are the obvious ones, while coverings of  $(U, T, \delta)$  are induced by Zariski covers of  $T$ . The structure sheaf  $\mathcal{O}_{S/\Sigma}$  of  $(S/\Sigma)_{\text{cris}}$  is defined by  $\mathcal{O}_{S/\Sigma}((U, T, \delta)) = \Gamma(T, \mathcal{O}_T)$ .

Given a  $\mathbb{Z}_p/p^e$ -algebra  $B$  and an ideal  $J \subset B$  endowed with divided powers  $\delta$ , the module of *differentials compatible with divided powers* is the quotient of the module of  $\Sigma$ -linear differentials by the relations  $d\delta_n(x) = \delta_{n-1}(x)d(x)$ , for  $x \in J$  and  $n \geq 1$ . We simply write  $\Omega_B^1$  for this module as confusion is unlikely. The formation of  $\Omega_B^1$  commutes with localisation on  $B$ , so the formula  $\Omega_{S/\Sigma}^1((U, T, \delta)) = \Gamma(T, \Omega_T^1)$  defines a sheaf  $\Omega_{S/\Sigma}^1$  on  $(S/\Sigma)_{\text{cris}}$ . Like its classical analogue, the sheaf  $\Omega_{S/\Sigma}^1$  can also be described via the diagonal as follows. Given an object  $(U, T, \delta)$  of  $(S/\Sigma)_{\text{cris}}$ , let  $(U, T(1), \delta(1))$  be the product of  $(U, T, \delta)$  with itself in  $(S/\Sigma)_{\text{cris}}$ : the scheme  $T(1)$  is simply the divided power envelope of  $U \subset T \times_{\Sigma} T$ , with  $\delta(1)$  being the induced divided power structure. The diagonal map  $\Delta : T \rightarrow T(1)$  is a closed immersion corresponding to an ideal sheaf  $\mathcal{I}$  with divided powers, and we have

$$\Omega_{S/\Sigma}^1((U, T, \delta)) = \Gamma(T, \mathcal{I}/\mathcal{I}^{[2]}),$$

where  $\mathcal{I}^{[2]}$  denotes the second divided power of  $\mathcal{I}$ . For  $i \geq 0$  we define  $\Omega_{S/\Sigma}^i$  as the  $i$ -th exterior power of  $\Omega_{S/\Sigma}^1$ .

An  $\mathcal{O}_{S/\Sigma}$ -module  $\mathcal{F}$  on  $(S/\Sigma)_{\text{cris}}$  is called *quasi-coherent* if for every object  $(U, T, \delta)$ , the restriction  $\mathcal{F}_T$  of  $\mathcal{F}$  to the Zariski site of  $T$  is a quasi-coherent  $\mathcal{O}_T$ -module. Examples include  $\mathcal{O}_{S/\Sigma}$  and  $\Omega_{S/\Sigma}^i$  for all  $i > 0$ .

An  $\mathcal{O}_{S/\Sigma}$ -module  $\mathcal{F}$  on  $(S/\Sigma)_{cris}$  is called a *crystal in quasi-coherent modules* if it is quasi-coherent and for every morphism  $f : (U, T, \delta) \rightarrow (U', T', \delta')$  the comparison map

$$c_f : f^* \mathcal{F}_{T'} \rightarrow \mathcal{F}_T$$

is an isomorphism. For example, the sheaf  $\mathcal{O}_{S/\Sigma}$  is a crystal (by fiat), but the sheaves  $\Omega_{S/\Sigma}^i, i > 0$  are **not** crystals.

Given a crystal  $\mathcal{F}$  in quasi-coherent modules and an object  $(U, T, \delta)$ , the projections define canonical isomorphisms

$$\mathrm{pr}_1^* \mathcal{F}_T \xrightarrow{c_1} \mathcal{F}_{T(1)} \xleftarrow{c_2} \mathrm{pr}_2^* \mathcal{F}_T.$$

These comparison maps are functorial in the objects of the crystalline site. Hence we obtain a canonical map

$$(1.0.1) \quad \nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^1$$

such that for any object  $(U, T, \delta)$  and any section  $s \in \Gamma(T, \mathcal{F}_T)$  we have

$$c_1(s \otimes 1) - c_2(1 \otimes s) = \nabla(s) \in \mathcal{I}/\mathcal{I}^{[2]} \otimes_{\mathcal{O}_{S/\Sigma}} \mathcal{F}_{T(1)}.$$

Transitivity of the comparison maps implies this connection is integrable, hence defines a *de Rham complex*

$$\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^1 \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^2 \rightarrow \cdots.$$

We remark that this complex does not terminate in general.

## 2. THE DE RHAM-CRYSTALLINE COMPARISON FOR AFFINES

In this section, we discuss the relationship between de Rham and crystalline cohomology (with coefficients) when  $S$  is affine. First, we establish some notation that will be used throughout this section.

**Notation 2.1.** Assume  $S = \mathrm{Spec}(A)$  for a  $\mathbf{Z}/p^N$ -algebra  $A$  (and some  $N > 0$ ). Choose a polynomial algebra  $P$  over  $\mathbf{Z}_p$  and a surjection  $P \rightarrow A$  with kernel  $J$ . Let  $D = D_J(P)^\wedge$  be the  $p$ -adically completed divided power envelope of  $P \rightarrow A$ . We set  $\Omega_D^1 := \Omega_P^1 \otimes_P^\wedge D$ , so  $\Omega_D^1/p^e \simeq \Omega_{D/p^e}^1$ . We also set  $D(0) = D$  and let

$$D(n) = D_{J(n)}(P \otimes_{\mathbf{Z}_p} \cdots \otimes_{\mathbf{Z}_p} P)^\wedge$$

where  $J(n) = \mathrm{Ker}(P \otimes \cdots \otimes P \rightarrow A)$  and where the tensor product has  $(n+1)$ -factors. For each  $e \geq N$  and any  $n \geq 0$ , we have a natural object  $(S, \mathrm{Spec}(D(n)/p^e(n)), \delta(n))$  of  $(S/\Sigma)_{cris}$ . Using this, for an abelian sheaf  $\mathcal{F}$  on  $(S/\Sigma)_{cris}$ , we define

$$\mathcal{F}(n) := \lim_{e \geq N} \mathcal{F}((S, \mathrm{Spec}(D(n)/p^e(n)), \delta(n))).$$

Each  $(S, \mathrm{Spec}(D(n)/p^e(n)), \delta(n))$  is simply the  $(n+1)$ -fold self-product of  $(S, \mathrm{Spec}(D/p^e), \delta)$  in  $(S/\Sigma)_{cris}$ . Letting  $n$  vary, we obtain a natural cosimplicial abelian group (or a cochain complex)

$$\mathcal{F}(\bullet) := (\mathcal{F}(0) \rightarrow \mathcal{F}(1) \rightarrow \mathcal{F}(2) \cdots),$$

that we call the *Čech-Alexander complex* of  $\mathcal{F}$  associated to  $D$ .

**2.2. Some generalities on crystalline cohomology.** This subsection collects certain basic tools necessary for working with crystalline cohomology; these will be used consistently in the sequel. We begin with a brief review of the construction of homotopy-limits in the only context where they appear in this paper.

**Construction 2.3.** Let  $\mathcal{C}$  be a topos. Fix a sequence  $T_1 \subset T_2 \subset \cdots \subset T_n \subset \cdots$  of monomorphisms in  $\mathcal{C}$ . We will construct the functor  $R \lim_i R\Gamma(T_i, -)$ ; here we follow the convention that  $\mathcal{G}(U) = \Gamma(U, \mathcal{G}) = \mathrm{Hom}_{\mathcal{C}}(U, \mathcal{G})$  for any pair of objects  $U, \mathcal{G} \in \mathcal{C}$ . Let  $\mathrm{Ab}^{\mathbf{N}}$  denote the category of projective systems of abelian groups indexed by the natural numbers. The functor  $\mathcal{F} \mapsto \lim_i \mathcal{F}(T_i)$  can be viewed as the composite

$$\mathrm{Ab}(\mathcal{C})^{\{\Gamma(T_i, -)\}_i} \mathrm{Ab}^{\mathbf{N}} \xrightarrow{\lim_i} \mathrm{Ab}.$$

Each of these functors is a left exact functor between abelian categories with enough injectives, so we obtain a composite of (triangulated) derived functors

$$D^+(\mathrm{Ab}(\mathcal{C}))^{\{R\Gamma(T_i, -)\}_i} D^+(\mathrm{Ab}^{\mathbf{N}}) \xrightarrow{R \lim_i} D^+(\mathrm{Ab}).$$

We use  $R \lim_i R\Gamma(T_i, -)$  to denote the composite functor. To identify this functor, observe that if we set  $T = \mathrm{colim}_i T_i$ , then  $\mathcal{F}(T) = \lim_i \mathcal{F}(T_i)$  by adjunction. Moreover, for any injective object  $\mathcal{I}$  of  $\mathrm{Ab}(\mathcal{C})$ , the projective system  $i \mapsto \mathcal{I}(T_i)$  has surjective transition maps  $\mathcal{I}(T_{i+1}) \rightarrow \mathcal{I}(T_i)$ : the maps  $T_i \rightarrow T_{i+1}$  are injective, and  $\mathcal{I}$  is an

injective object. Since projective systems in  $\text{Ab}^{\mathbf{N}}$  with surjective transition maps are acyclic for the functor  $\lim_i$  (by the Mittag-Leffler condition), there is an identification of triangulated functors

$$R\Gamma(T, -) \simeq R\lim_i R\Gamma(T_i, -).$$

Thus, the value  $R\lim_i R\Gamma(T_i, \mathcal{F})$  is computed by  $I^\bullet(T) = \lim_i I^\bullet(T_i)$ , where  $\mathcal{F} \rightarrow I^\bullet$  is an injective resolution. An observation that will be useful in the sequel is the following: if each  $R\Gamma(T_i, \mathcal{F})$  is concentrated in degree 0, then  $R\lim_i R\Gamma(T_i, \mathcal{F})$  coincides with  $R\lim_i \mathcal{F}(T_i)$ , and thus has only two non-zero cohomology groups (as  $R^j \lim_i A_i = 0$  for  $j > 1$  and any  $\mathbf{N}$ -indexed projective system  $\{A_i\}_i$  of abelian groups).

We use Construction 2.3 to show that the Čech-Alexander complex often computes crystalline cohomology (compare with [Ber74, Theorem V.1.2.5]).

**Lemma 2.4.** *Let  $\mathcal{F}$  be a quasi-coherent  $\mathcal{O}_{S/\Sigma}$ -module. Assume that for each  $n > 0$ , the group  $R^1 \lim_{e \geq N} \mathcal{F}(S, \text{Spec}(D(n)/p^e D(n)), \delta(n))$  vanishes for the projective system  $e \mapsto \mathcal{F}((S, \text{Spec}(D(n)/p^e D(n)), \delta(n)))$ . Then the complex  $\mathcal{F}(\bullet)$  computes  $R\Gamma(S/\Sigma, \mathcal{F})$ .*

*Proof.* As representable functors are sheaves on  $(S/\Sigma)_{\text{cris}}$  (by Zariski descent), we freely identify objects of  $(S/\Sigma)_{\text{cris}}$  with the corresponding sheaf on  $(S/\Sigma)_{\text{cris}}$ . One can easily check that the map  $\text{colim}_{e \geq N} \left( (S, \text{Spec}(D/p^e), \delta) \right) \rightarrow *$  is an effective epimorphism in the topos of sheaves on  $(S/\Sigma)_{\text{cris}}$ . Since filtered colimits and the Yoneda embedding both commute with finite products, the  $(n+1)$ -fold self-product of  $\text{colim}_{e \geq N} \left( (S, \text{Spec}(D/p^e), \delta) \right)$  is simply  $\text{colim}_{e \geq N} \left( (S, \text{Spec}(D(n)/p^e(n)), \delta(n)) \right)$ . General topos theory (see Remark 2.5) shows that  $R\Gamma(S/\Sigma, \mathcal{F})$  is computed by

$$R\Gamma(\text{colim}_{e \geq N} \left( (S, \text{Spec}(D(0)/p^e(0)), \delta(0)) \right), \mathcal{F}) \rightarrow R\Gamma(\text{colim}_{e \geq N} \left( (S, \text{Spec}(D(1)/p^e(1)), \delta(1)) \right), \mathcal{F}) \rightarrow \dots$$

The discussion in Construction 2.3 and the vanishing of quasi-coherent sheaf cohomology on affine schemes then identify the above bicomplex with the bicomplex

$$R\lim_{e \geq N} (\mathcal{F}((S, \text{Spec}(D(0)/p^e D(0)), \delta(0)))) \rightarrow R\lim_{e \geq N} (\mathcal{F}((S, \text{Spec}(D(1)/p^e D(1)), \delta(1)))) \rightarrow \dots$$

The  $R^1 \lim_e$  vanishing hypothesis ensures that the bicomplex above collapses to  $\mathcal{F}(\bullet)$  proving the claim.  $\square$

**Remark 2.5.** The following fact was used in the proof of Lemma 2.4: if  $\mathcal{C}$  is a topos, and  $X \rightarrow *$  is an effective epimorphism in  $\mathcal{C}$ , then for any abelian sheaf  $\mathcal{F}$  in  $\mathcal{C}$ , the object  $R\Gamma(*, \mathcal{F})$  is computed by a bicomplex

$$R\Gamma(X, \mathcal{F}) \rightarrow R\Gamma(X \times X, \mathcal{F}) \rightarrow R\Gamma(X \times X \times X, \mathcal{F}) \rightarrow \dots,$$

i.e., the choice of an injective resolution  $\mathcal{F} \rightarrow I^\bullet$  defines a bicomplex  $I^\bullet(X^{\bullet+1})$  whose totalisation computes  $R\Gamma(*, \mathcal{F})$ ; this follows from cohomological descent since the augmented simplicial object  $X^\bullet \rightarrow *$  is a hypercover. In particular, there is a spectral sequence with  $E_1$ -term given by  $H^q(X^p, \mathcal{F})$  that converges to  $H^{p+q}(*, \mathcal{F})$ .

**Remark 2.6.** The  $R^1 \lim_{e \geq N}$  vanishing assumption of Lemma 2.4 will hold for all sheaves appearing in this paper. For quasi-coherent crystals  $\mathcal{F}$ , this assumption clearly holds as

$$\mathcal{F}((S, \text{Spec}(D(n)/p^e(n)), \delta(n))) \rightarrow \mathcal{F}((S, \text{Spec}(D(n)/p^{e-1}(n)), \delta(n)))$$

is surjective for all  $e > N$  and all  $n \geq 0$ . By direct computation, the same is also true for the sheaves  $\Omega_{S/\Sigma}^i$ .

Next, we formulate and prove a purely algebraic lemma comparing  $p$ -adically complete  $\mathbf{Z}_p$ -modules with compatible systems of  $\mathbf{Z}/p^e$ -modules; the result is elementary and well-known, but recorded here for convenience. We remind the reader that a  $\mathbf{Z}_p$ -module  $M$  is said to be  *$p$ -adically complete* if the natural map  $M \rightarrow \lim_e M/p^e$  is an isomorphism.

**Lemma 2.7.** *The functor  $M \mapsto (M/p^e, \text{can}_e : (M/p^{e+1})/p^e \simeq M/p^e)$  defines an equivalence between the category of  $p$ -adically complete  $\mathbf{Z}_p$ -modules  $M$  and the category of projective systems  $(M_e, \phi_e)$  (indexed by  $e \in \mathbf{N}$ ) with  $M_e$  a  $\mathbf{Z}/p^e$ -module, and  $\phi_e : M_{e+1}/p^e \simeq M_e$  an isomorphism.*

*Proof.* A left-inverse functor is given by  $(M_e, \phi_e) \mapsto M := \lim_e M_e$ , the limit being taken along the maps  $\phi_e$ . To check that this is also a right inverse, it suffices to show that  $(\lim_e M_e)/p^n \simeq M_n$  for any system  $(M_e, \phi_e)$  as in the lemma. Projection defines a natural map  $(\lim_e M_e)/p^n \rightarrow M_n$  which is surjective as the  $\phi_e$ 's are all surjective. For injectivity, it suffices to show that any element  $m = (m_e) \in \lim_e M_e$  with  $m_n = 0$  is divisible by  $p^n$  in  $\lim_e M_e$ . The hypothesis implies that there exists an  $m' \in \lim_e M_e$  such that  $m - p^n m'$  maps to 0 in  $M_{n+1}$ : we can simply take  $m'$

to be an arbitrary lift of an element in  $M_{n+1}$  which gives  $m_{n+1}$  on multiplication by  $p^n$  (which exists since  $\phi_{e+1}$  maps  $M_{n+1}/p^n$  isomorphically onto  $M_n$ ). Continuing this process, for each  $i > 0$ , we can find an element  $m_i \in \lim_e M_e$  such that  $m - p^n m_i$  maps to 0 in  $M_{n+i}$ . Taking the limit  $i \rightarrow \infty$  proves the desired claim.  $\square$

The next lemma is a standard result in crystalline cohomology (see [Ber74, Chapter IV] and [BO78, Theorem 6.6]). We sketch the proof to convince the reader that this result is elementary.

**Lemma 2.8.** *The category of crystals in quasi-coherent  $\mathcal{O}_{S/\Sigma}$ -modules is equivalent to the category of pairs  $(M, \nabla)$  where  $M$  is a  $p$ -adically complete  $D$ -module and  $\nabla : M \rightarrow M \otimes_D^{\wedge} \Omega_D^1$  is a topologically quasi-nilpotent integrable connection.*

*Proof.* Given a crystal in quasi-coherent modules  $\mathcal{F}$  we set

$$M = \mathcal{F}(0) := \lim_{e \geq N} \mathcal{F}((S, \text{Spec}(D/p^e D), \delta))$$

and  $\nabla$  is as in (1.0.1). Conversely, suppose that  $(M, \nabla)$  is a module with connection as in the statement of the lemma. Then, given an affine object  $(S \hookrightarrow T, \delta)$  of the crystalline site corresponding to the divided power thickening  $(B \rightarrow A, \delta)$ , we set

$$\mathcal{F}((S, T, \delta)) = M \otimes_D B$$

where  $D \rightarrow B$  is any divided power map lifting  $\text{id}_A : A \rightarrow A$ . Note that  $p^m B = 0$  for some  $m \geq 0$  by the definition of the crystalline site, so completion isn't needed in the formula. To see that this is well defined suppose that  $\varphi_1, \varphi_2 : D \rightarrow B$  are two maps lifting  $\text{id}_A$ . Then we have an isomorphism

$$M_{\varphi_1, \varphi_2} : M \otimes_{D, \varphi_1} B \longrightarrow M \otimes_{D, \varphi_2} B$$

which is  $B$ -linear and characterized by the (Taylor) formula

$$m \otimes 1 \longmapsto \sum_{E=(e_i)} \left( \prod (\nabla_{\vartheta_i})^{e_i} \right) (m) \otimes \prod \delta_{e_i}(h_i)$$

where the sum is over all multi-indices  $E$  with finite support. The notation here is:  $P = \mathbf{Z}_p[\{x_i\}_{i \in I}]$ ,  $\vartheta_i = \partial/\partial x_i$  and  $h_i = \varphi_2(x_i) - \varphi_1(x_i)$ . Since  $h_i \in \text{Ker}(B \rightarrow A)$  it makes sense to apply the divided powers  $\delta_e$  to  $h_i$ . The sum converges precisely because the connection is topologically quasi-nilpotent (this can be taken as the definition). For three maps  $\varphi_1, \varphi_2, \varphi_3 : D \rightarrow B$  lifting  $\text{id}_A$ , the resulting isomorphisms satisfy the cocycle condition

$$M_{\varphi_2, \varphi_3} \circ M_{\varphi_1, \varphi_2} = M_{\varphi_1, \varphi_3}$$

by the flatness of  $\nabla$ . Hence, the above recipe defines a sheaf on  $(S/\Sigma)_{\text{cris}}$ .  $\square$

**Remark 2.9.** Lemma 2.8 remains valid if we replace the polynomial algebra  $P$  appearing in Notation 2.1 with any smooth  $\mathbf{Z}_p$ -algebra  $P$  equipped with a surjection to  $A$  (and  $D$  with the corresponding  $p$ -adically completed divided power envelope). The only non-obvious point is to find a replacement for the Taylor series appearing in the formula for  $M_{\varphi_1, \varphi_2}$  in the proof of Lemma 2.8. However, note first that the Taylor series makes sense as soon as there is a polynomial algebra  $F$  and an étale map  $F \rightarrow P$ . Moreover, a ‘‘change of variables’’ computation shows that the resulting map is independent of choice of étale chart  $F \rightarrow P$ . The general case then follows by Zariski glueing.

We use Lemma 2.8 to show that crystals on  $(S/\Sigma)_{\text{cris}}$  are determined by their restriction to the special fibre.

**Corollary 2.10.** *Reduction modulo  $p$  gives an equivalence between categories of crystals of quasi-coherent modules over the structure sheaves of  $(S/\Sigma)_{\text{cris}}$  and  $(S \otimes_{\Sigma} \text{Spec}(\mathbf{F}_p)/\Sigma)_{\text{cris}}$*

*Proof.* This follows from Lemma 2.8 by identifying, for any  $e \geq N$ , the divided power envelopes of  $P/p^e \rightarrow A$  and  $P/p^e \rightarrow A \rightarrow A/p$  (by [Ber74, Proposition I.2.8.2]).  $\square$

**2.11. The main theorem in the affine case.** The goal of this section is to prove the following theorem.

**Theorem 2.12.** *Suppose that  $\mathcal{F}$  corresponds to a pair  $(M, \nabla)$  as in Lemma 2.8. Then there is a natural quasi-isomorphism*

$$R\Gamma(S/\Sigma, \mathcal{F}) \simeq (M \rightarrow M \otimes_D^{\wedge} \Omega_D^1 \rightarrow M \otimes_D^{\wedge} \Omega_D^2 \rightarrow \dots).$$

For  $\mathcal{F}$  and  $M$  as in Theorem 2.12, we use  $M(n)$  and  $M(\bullet)$  instead of  $\mathcal{F}(n)$  and  $\mathcal{F}(\bullet)$  from Notation 2.1. Each  $M(n)$  is a  $D(n)$ -module with integrable connection as in (1.0.1), so it defines a de Rham complex

$$M(n) \rightarrow M(n) \otimes_{D(n)}^{\wedge} \Omega_{D(n)}^1 \rightarrow M(n) \otimes_{D(n)}^{\wedge} \Omega_{D(n)}^2 \rightarrow \cdots$$

As  $n$  varies, these complexes fit together to define a bicomplex, which we call the de Rham complex of  $M(\bullet)$ . Our proof of Theorem 2.12 hinges on the observation that each side of the quasi-isomorphism occurring in the statement of Theorem 2.12 also appears in the de Rham complex of  $M(\bullet)$ : the left side is the 0-th row, while the right side is the 0-th column. Thus, the proof of Theorem 2.12 is reduced to certain acyclicity results for the de Rham complex of  $M(\bullet)$ , which we show next. The following lemma shows that the ‘‘columns’’ of this bicomplex are all quasi-isomorphic.

**Lemma 2.13.** *The map of complexes*

$$M \otimes_D^{\wedge} \Omega_D^* \rightarrow M(n) \otimes_{D(n)}^{\wedge} \Omega_{D(n)}^*$$

induced by any of the natural maps  $D \rightarrow D(n)$  is a quasi-isomorphism.

*Proof.* This is the ‘‘naive’’ Poincaré lemma. More precisely, each natural map  $D \rightarrow D(n)$  defines an isomorphism of  $D(n)$ -modules  $M(n) \simeq M \otimes_D^{\wedge} D(n)$  compatible with  $\nabla$  by the crystalline nature of  $\mathcal{F}$ . Thus, there is a filtration of  $M(n) \otimes_{D(n)}^{\wedge} \Omega_{D(n)}^*$  whose graded pieces are  $M \otimes_D^{\wedge} \Omega_D^i \otimes_D^{\wedge} \Omega_{D(n)/D}^*$ . Thus it suffices to show the natural map

$$D \rightarrow \left( D(n) \rightarrow \Omega_{D(n)/D}^1 \rightarrow \Omega_{D(n)/D}^2 \rightarrow \cdots \right)$$

is a quasi-isomorphism. This can be checked explicitly as  $D(n)$  is a divided power polynomial algebra over  $D$  (see [Ber74, Lemma V.2.1.2]).  $\square$

Next, we identify the ‘‘rows’’ of the de Rham complex of  $M(\bullet)$ .

**Lemma 2.14.** *The complex*

$$M \otimes_D^{\wedge} \Omega_D^i \rightarrow M(1) \otimes_{D(1)}^{\wedge} \Omega_{D(1)}^i \rightarrow M(2) \otimes_{D(2)}^{\wedge} \Omega_{D(2)}^i \cdots$$

computes  $R\Gamma(S/\Sigma, \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^i)$ .

*Proof.* By Lemma 2.7, we have

$$M \otimes_D^{\wedge} \Omega_D^i \simeq \lim_{e \geq N} \left( M/p^e \otimes_{D/p^e D} \Omega_{D/p^e D}^i \right) \simeq \lim_{e \geq N} \left( (\mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^i)((S, \text{Spec}(D/p^e), \delta)) \right),$$

and similarly for the terms over  $D(n)$ . The claim now follows from Lemma 2.4, the fact that  $\Omega_{S/\Sigma}^i$  is quasi-coherent, and the fact that the transition maps  $M/p^{e+1} \otimes_{D/p^{e+1} D} \Omega_{D/p^{e+1} D}^i \rightarrow M/p^e \otimes_{D/p^e D} \Omega_{D/p^e D}^i$  are surjective.  $\square$

To finish the proof of Theorem 2.12, we need an acyclicity result about the ‘‘rows’’ of the de Rham complex of  $M(\bullet)$ . First, we handle the case  $M = D$ , i.e., when  $\mathcal{F} = \mathcal{O}_{S/\Sigma}$ .

**Lemma 2.15.** *The complex*

$$\Omega_D^1 \rightarrow \Omega_{D(1)}^1 \rightarrow \Omega_{D(2)}^1 \rightarrow \cdots$$

is homotopic to zero as a  $D(\bullet)$ -cosimplicial module.

*Proof.* This complex is equal to the base change of the cosimplicial module

$$M_* = \left( \Omega_P^1 \rightarrow \Omega_{P \otimes P}^1 \rightarrow \Omega_{P \otimes P \otimes P}^1 \rightarrow \cdots \right)$$

via the cosimplicial ring map  $P^{\otimes n+1} \rightarrow D(n)$ . Hence it suffices to show that the cosimplicial module  $M_*$  is homotopic to zero. Let  $P = \mathbf{Z}_p[\{x_i\}_{i \in I}]$ . Then  $P^{\otimes n+1}$  is the polynomial algebra on the elements

$$x_i(e) = 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1$$

with  $x_i$  in the  $e$ th slot. The modules of the complex are free on the generators  $dx_i(e)$ . Note that if  $f : [n] \rightarrow [m]$  is a map then we see that

$$M_*(f)(dx_i(e)) = dx_i(f(e))$$

Hence we see that  $M_*$  is a direct sum of copies of Example 2.16 indexed by  $I$ , and we win.  $\square$

**Example 2.16.** Suppose that  $A_*$  is any cosimplicial ring. Consider the cosimplicial module  $M_*$  defined by the rule

$$M_n = \bigoplus_{i=0, \dots, n} A_n e_i$$

For a map  $f : [n] \rightarrow [m]$  define  $M_*(f) : M_n \rightarrow M_m$  to be the unique  $A_*(f)$ -linear map which maps  $e_i$  to  $e_{f(i)}$ . We claim the identity on  $M_*$  is homotopic to 0. Namely, a homotopy is given by a map of cosimplicial modules

$$h : M_* \longrightarrow \text{Hom}(\Delta[1], M_*)$$

where  $\Delta[1]$  denote the simplicial set whose set of  $n$ -simplices is  $\text{Mor}([n], [1])$ , see [Mey90]. Let  $\alpha_j^n : [n] \rightarrow [1]$  be defined by  $\alpha_j^n(i) = 0 \Leftrightarrow i < j$ . Then we define  $h$  in degree  $n$  by the rule

$$h_n(e_i)(\alpha_j^n) = \begin{cases} e_i & \text{if } i < j \\ 0 & \text{else} \end{cases}$$

We first check  $h$  is a morphism of cosimplicial modules. Namely, for  $f : [n] \rightarrow [m]$  we will show that

$$(2.16.1) \quad h_m \circ M_*(f) = \text{Hom}(\Delta[1], M_*)(f) \circ h_n$$

This is equivalent to saying that the left hand side of (2.16.1) evaluated at  $e_i$  is given by

$$h(e_{f(i)})(\alpha_j^m) = \begin{cases} e_{f(i)} & \text{if } f(i) < j \\ 0 & \text{else} \end{cases}$$

Note that  $\alpha_j^m \circ f = \alpha_{j'}^n$  where  $0 \leq j' \leq n$  is such that  $f(a) < j$  if and only if  $a < j'$ . Thus the right hand side of (2.16.1) evaluated at  $e_i$  is given by

$$M_*(f)(h(e_i)(\alpha_j^m \circ f)) = M_*(f)(h(e_i)(\alpha_{j'}^n)) = \begin{cases} e_{f(i)} & \text{if } i < j' \\ 0 & \text{else} \end{cases}$$

It follows from our description of  $j'$  that the two answers are equal. Hence  $h$  is a map of cosimplicial modules. Let  $0 : \Delta[0] \rightarrow \Delta[1]$  and  $1 : \Delta[0] \rightarrow \Delta[1]$  be the obvious maps, and denote  $ev_0, ev_1 : \text{Hom}(\Delta[1], M_*) \rightarrow M_*$  the corresponding evaluation maps. The reader verifies readily that the the compositions

$$ev_0 \circ h, ev_1 \circ h : M_* \longrightarrow M_*$$

are 0 and 1 respectively, whence  $h$  is the desired homotopy between 0 and 1.

We now extend Lemma 2.15 to allow non-trivial coefficients.

**Lemma 2.17.** *For all  $i > 0$  cosimplicial module*

$$M \otimes_D^{\wedge} \Omega_D^i \rightarrow M(1) \otimes_{D(1)}^{\wedge} \Omega_{D(1)}^i \rightarrow M(2) \otimes_{D(2)}^{\wedge} \Omega_{D(2)}^i \cdots$$

*is homotopy equivalent to zero.*

*Proof.* The cosimplicial  $D(\bullet)$ -module above is a (termwise) completed tensor product of the cosimplicial  $D(\bullet)$ -modules  $M(\bullet)$  and  $\Omega_{D(\bullet)}^i$ . Lemma 2.15 shows that  $\Omega_{D(\bullet)}^1$  is homotopy equivalent to zero as a cosimplicial  $D(\bullet)$ -module. The claim now follows as the following three operations preserve the property of being homotopy equivalent to 0 for cosimplicial  $D(\bullet)$ -modules: termwise application of  $\wedge^i$ , tensoring with another cosimplicial  $D(\bullet)$ -module, and termwise  $p$ -adic completion.  $\square$

The material above gives a rather pleasing proof that crystalline cohomology is computed by the de Rham complex:

*Proof of Theorem 2.12.* We look at the first quadrant double complex  $M^{\bullet, \bullet}$  with terms

$$M^{n, m} = M(n) \otimes_{D(n)}^{\wedge} \Omega_{D(n)}^m.$$

The horizontal differentials are given determined by the Čech-Alexander complex, while the vertical ones are given by the de Rham complex. By Lemma 2.13, each column complex  $M^{n, \bullet}$  is quasi-isomorphic to de Rham complex  $M \otimes_D^{\wedge} \Omega_D^{\bullet}$ . Hence  $H^m(M^{n, \bullet})$  is independent of  $n$  and the differentials are

$$H^m(M^{0, \bullet}) \xrightarrow{0} H^m(M^{1, \bullet}) \xrightarrow{1} H^m(M^{2, \bullet}) \xrightarrow{0} H^m(M^{3, \bullet}) \xrightarrow{1} \cdots$$

We conclude that  $\text{Tot}(M^{\bullet, \bullet})$  computes the cohomology of the de Rham complex  $M \otimes_D^{\wedge} \Omega_D^*$  by the first spectral sequence associated to the double complex. On the other hand, Lemma 2.4 shows that the “row” complex  $M^{\bullet, 0}$  computes the cohomology of  $\mathcal{F}$ . Hence if we can show that each  $M^{\bullet, m}$  for  $m > 0$  is acyclic, then we’re done by the second spectral sequence. The desired vanishing now follows from Lemma 2.17.  $\square$

**Remark 2.18.** Lemma 2.8 and Theorem 2.12 remain valid when  $D$  is taken to be the  $p$ -adic completion of the divided power envelope of a surjection  $P \rightarrow A$  with  $P$  any smooth  $\mathbf{Z}_p$ -algebra. For Lemma 2.8, this was discussed in Remark 2.9. Thus, the only non-obvious point now is whether an analogue of Lemma 2.15 is valid. However, at least Zariski locally on  $\mathrm{Spec}(P)$ , there is an étale map  $F \rightarrow P$  with  $F$  a polynomial  $\mathbf{Z}_p$ -algebra. Thus, the cotangent bundle of  $P$  (and hence that of  $D$ ) is obtained by base change from that of  $F$ , so the required claim follows from the proof of Lemma 2.15. This shows that the assertion of Theorem 2.12 is true Zariski locally on  $S$ , and hence globally by the Čech spectral sequence for a suitable affine cover.

**Remark 2.19.** Let  $\Sigma_e = \mathrm{Spec}(\mathbf{Z}/p^e)$ , and let  $S$  be an affine  $\Sigma_e$ -scheme. One can define the crystalline site  $(S/\Sigma_e)_{\mathrm{cris}}$  and crystals in the obvious way. The arguments given in this section work *mutatis mutandis* to show that the cohomology  $R\Gamma(S/\Sigma_e, \mathcal{F})$  of a crystal  $\mathcal{F}$  of quasi-coherent  $\mathcal{O}_{S/\Sigma_e}$ -modules is computed by the de Rham complex

$$M_e \rightarrow M_e \otimes_{D/p^e} \Omega_{D/p^e}^1 \rightarrow M_e \otimes_{D/p^e} \Omega_{D/p^e}^2 \cdots,$$

where  $D/p^e$  is as in Notation 2.1, and  $M_e = \mathcal{F}((S, \mathrm{Spec}(D/p^e), \delta))$  is the  $D/p^e$ -module that is the value of the crystal  $\mathcal{F}$  on  $\mathrm{Spec}(D/p^e)$ , equipped with the integrable connection as in (1.0.1).

### 3. GLOBAL ANALOGUES

Our goal in this section is to prove a global analogue (Theorem 3.6) of the results of §2, and deduce some geometric consequences (Corollaries 3.8 and 3.10). In order to do so, we first conceptualize the work done in §2 as a *vanishing* result on arbitrary schemes in Theorem 3.2; this formulation gives us direct access to certain globally defined maps, which are then used to effortlessly reduce global statements to local ones.

**3.1. A vanishing statement.** Our vanishing result is formulated terms of the “change of topology” map relating the crystalline site to the Zariski site, whose construction we recall first. Let  $f : S \rightarrow \Sigma$  be a map with  $p$  locally nilpotent on  $S$ . There is a morphism of ringed topoi

$$u_{S/\Sigma} : \left( \mathrm{Shv}((S/\Sigma)_{\mathrm{cris}}), \mathcal{O}_{S/\Sigma} \right) \rightarrow \left( \mathrm{Shv}(S_{\mathrm{zar}}), f^{-1}\mathcal{O}_\Sigma \right)$$

characterised by the formula

$$u_{S/\Sigma}^{-1}(F)((U, T, \delta)) = F(U)$$

for any sheaf  $F \in \mathrm{Shv}(S_{\mathrm{zar}})$  and object  $(U, T, \delta) \in (S/\Sigma)_{\mathrm{cris}}$  (see [Ber74, §III.3.2]). The associated pushforward  $Ru_{S/\Sigma} : D(\mathcal{O}_{S/\Sigma}) \rightarrow D(f^{-1}\mathcal{O}_\Sigma)$  is a localised version of crystalline cohomology, i.e., for any  $U \in S_{\mathrm{zar}}$ , we have

$$R\Gamma(U, Ru_{S/\Sigma}(F)) \simeq R\Gamma(U/\Sigma, F);$$

see [Ber74, Corollary III.3.2.4] for the corresponding statement at the level of cohomology groups. With this language, our main result is the following somewhat surprising theorem.

**Theorem 3.2.** *Let  $S$  be a scheme over  $\Sigma$  such that  $p$  is locally nilpotent on  $S$ . Let  $\mathcal{F}$  be a crystal in quasi-coherent  $\mathcal{O}_{S/\Sigma}$ -modules. The truncation map of complexes*

$$(\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^1 \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^2 \rightarrow \cdots) \longrightarrow \mathcal{F}[0],$$

*while not a quasi-isomorphism, becomes a quasi-isomorphism after applying  $Ru_{S/\Sigma}$ . In fact, for any  $i > 0$ , we have*

$$Ru_{S/\Sigma}(\mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^i) = 0.$$

*Proof.* This follows from the vanishing of the cohomology of the sheaves  $\mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^i$  over affines for  $i > 0$ , see Lemmas 2.14 and 2.17.  $\square$

**Remark 3.3.** The *proof* of Theorem 3.2 shows that the conclusion  $Ru_{S/\Sigma}(\mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^i) = 0$  for  $i > 0$  is true for any quasi-coherent  $\mathcal{O}_{S/\Sigma}$ -module  $\mathcal{F}$  which, locally on  $S$ , satisfies the  $R^1 \lim_{e \geq N} \varphi_e$  vanishing condition of Lemma 2.4. This applies to the following non-crystals:  $\Omega_{S/\Sigma}^i$  for all  $i$ , and any sheaf of the form  $u_{S/\Sigma}^{-1}\mathcal{F}$ , where  $\mathcal{F}$  is a quasi-coherent  $\mathcal{O}_S$ -module on  $S_{\mathrm{zar}}$ . In particular, it applies to the sheaf  $u_{S/\Sigma}^{-1}\mathcal{O}_S$  defined by  $u_{S/\Sigma}^{-1}\mathcal{O}_S((U, T, \delta)) = \Gamma(U, \mathcal{O}_U)$ .

**3.4. Global results.** We now explain how to deduce global consequences from Theorem 3.2, such as the identification of crystalline cohomology with de Rham cohomology. First, we establish notation used in this section.

**Notation 3.5.** Let  $S$  be a  $\Sigma$ -scheme such that  $p^N = 0$  on  $S$ . Assume there is a closed immersion  $i : S \rightarrow X$  of  $\Sigma$ -schemes with  $X$  finitely presented and smooth over  $\Sigma$ . For each  $e \geq N$ , set  $\mathcal{D}_e$  to be the divided power hull of the map  $\mathcal{O}_X/p^e \rightarrow \mathcal{O}_S$ . Each  $\mathcal{D}_e$  is supported on  $S$ , and letting  $e$  vary defines a  $p$ -adic formal scheme  $T$  with underlying space  $S$  and structure sheaf  $\lim_{e \geq N} \mathcal{D}_e$ . Moreover, the category of quasi-coherent  $\mathcal{O}_T$ -modules can be identified with the category of compatible systems of  $\mathcal{D}_e$ -modules on  $S$  (with compatibilities as in Lemma 2.7), and we only use  $T$  as a tool for talking about such compatible systems. The sheaves  $\mathcal{D}_e$  define (honest) subschemes  $T_e = \text{Spec}(\mathcal{D}_e) \subset T$  containing  $S$ . The quasi-compactness of  $S$  then gives objects  $(S, T_e, \delta)$  of  $(S/\Sigma)_{\text{cris}}$ . Following our conventions, let  $\Omega_{T_e}^i$  be the pullback of the corresponding sheaf on  $X$ , and set  $\Omega_T^i$  to be the result of gluing the  $\Omega_{T_e}^i$ .

Let  $S$ ,  $T_e$ , and  $T$  be as in Notation 3.5, and let  $\mathcal{F}$  be a crystal in quasi-coherent  $\mathcal{O}_{S/\Sigma}$ -modules. Restricting  $\mathcal{F}$  to  $(S, T_e, \delta)$  defines quasi-coherent  $\mathcal{O}_{T_e}$ -modules  $\mathcal{F}_{T_e}$  for  $e \geq N$ , and hence a quasi-coherent  $\mathcal{O}_T$ -module  $\mathcal{M}$  by gluing. The integrable connections  $\mathcal{F}_{T_e} \rightarrow \mathcal{F}_{T_e} \otimes_{\mathcal{O}_{T_e}} \Omega_{T_e}^1$  coming from (1.0.1) glue to give an integrable connection

$$\nabla : \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^1,$$

which then defines a de Rham complex on  $T$ .

**Theorem 3.6.** *Let  $S \rightarrow X$ ,  $T$ ,  $\mathcal{F}$ , and  $\mathcal{M}$  be as in Notation 3.5 and the following discussion. The hypercohomology on  $T$  of the complex*

$$\mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^1 \rightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^2 \rightarrow \dots$$

*computes  $R\Gamma(S/\Sigma, \mathcal{F})$ .*

*Proof.* First, we construct the map. By basic formal scheme theory, we have a formula

$$\begin{aligned} R \lim_{e \geq N} R\Gamma((S, T_e, \delta), \mathcal{F} \otimes_{\mathcal{O}_{S/\Sigma}} \Omega_{S/\Sigma}^i) &= R \lim_{e \geq N} R\Gamma(T_e, \mathcal{F}_{T_e} \otimes_{\mathcal{O}_{T_e}} \Omega_{T_e}^i) \\ &= R\Gamma(T, R \lim_{e \geq N} (\mathcal{F}_{T_e} \otimes_{\mathcal{O}_{T_e}} \Omega_{T_e}^i)) \\ &= R\Gamma(T, \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^i) \end{aligned}$$

for each  $i \geq 0$ . Here the first equality follows from the definition of cohomology in the crystalline site; the second equality follows from the identification of the (derived functors of the) composite functors

$$\text{Ab}(\mathcal{C})^{\mathbf{N}} \xrightarrow{\Gamma(T, -)} \text{Ab}^{\mathbf{N}} \xrightarrow{\lim} \text{Ab} \quad \text{and} \quad \text{Ab}(\mathcal{C})^{\mathbf{N}} \xrightarrow{\lim} \text{Ab}(\mathcal{C}) \xrightarrow{\Gamma(T, -)} \text{Ab};$$

the last equality follows from the vanishing of

$$R^j \lim_{e \geq N} (\mathcal{F}_e \otimes_{\mathcal{O}_{T_e}} \Omega_{T_e}^i)$$

for all  $j > 0$ , which follows from the vanishing of higher quasi-coherent sheaf cohomology for affines. Using this formula, and applying

$$R\Gamma(*, -) \longrightarrow R \lim_{e \geq N} R\Gamma((S, T_e, \delta), -)$$

to the morphism in Theorem 3.2, gives the desired map

$$R\Gamma(S/\Sigma, \mathcal{F}) \rightarrow R\Gamma(T, \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^1 \rightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^2 \rightarrow \dots).$$

Moreover, this map is an isomorphism for affine  $S$  by Theorem 2.12 (see Remark 2.18), and is functorial in  $S$ . The Čech spectral sequence for an affine open cover then immediately implies the claim for  $X$  is quasi-compact and separated (as the  $E_2$  terms involve cohomology of affines). For an  $X$  only assumed to be quasi-compact and quasi-separated, another application of the spectral sequence for an affine open cover finishes the proof (as the  $E_2$  terms involve cohomology on quasi-compact and separated schemes). Since all smooth finitely presented  $\Sigma$ -schemes are quasi-compact and quasi-separated, we are done.  $\square$

**Remark 3.7.** The arguments that go into proving Theorem 3.6 also apply *mutatis mutandis* to reprove Grothendieck's comparison theorem from [Gro68]: if  $S$  is a variety over  $\mathbf{C}$ , and  $S \subset X$  is a closed immersion into a smooth variety, and  $T$  denotes the formal completion of  $X$  along  $S$ , then the cohomology of the structure sheaf on the infinitesimal site  $(S/\text{Spec}(\mathbf{C}))_{\text{inf}}$  is computed by the hypercohomology on  $S$  of the de Rham complex of  $T$  (defined suitably). The only essential change is that the proof of Lemma 2.13, which relies on the vanishing of the higher de Rham

cohomology of a divided power polynomial algebra, must be replaced by its formal analogue, i.e., the vanishing of the higher de Rham cohomology of a formal power series ring in characteristic 0.

In certain situations, Theorem 3.6 can be algebraised to get a statement about classical schemes. For example:

**Corollary 3.8.** *Let  $f : X \rightarrow \Sigma$  be a proper smooth morphism, and set  $S = X \times_{\Sigma} \text{Spec}(\mathbf{F}_p)$ . Then the hypercohomology of the de Rham complex*

$$\mathcal{O}_X \rightarrow \Omega_{X/\Sigma}^1 \rightarrow \Omega_{X/\Sigma}^2 \rightarrow \cdots,$$

*computes  $R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma})$ . In particular, the de Rham cohomology of  $X \rightarrow \Sigma$  is determined functorially by the fibre  $S \hookrightarrow X$  of  $f$ , and thus admits a Frobenius action.*

*Proof.* This follows from Theorem 3.6 and the formal functions theorem as  $T$  is just the  $p$ -adic completion of  $X$ : the ideal  $\ker(\mathcal{O}_X \rightarrow \mathcal{O}_S) = (p) \subset \mathcal{O}_X$  already has specified divided powers, so  $\mathcal{O}_X/p^e = \mathcal{D}_e$  for any  $e$ .  $\square$

**Remark 3.9.** One can upgrade Corollary 3.8 to a statement that incorporates coefficients as follows. There is an equivalence of categories between crystals in quasi-coherent sheaves on  $(S/\Sigma)$  and quasi-coherent sheaves on  $X$  equipped with a flat connection relative to  $\Sigma$  (see Lemma 2.8 and Corollary 2.10). This equivalence respects cohomology, i.e., the crystalline cohomology of a crystal in quasi-coherent sheaves on  $S/\Sigma$  is computed by the de Rham cohomology of the corresponding quasi-coherent sheaf with flat connection on  $X$ .

We conclude with a brief discussion of the base change behaviour. For an  $\mathbf{Z}/p^N$ -scheme  $S$  as above, a natural question is whether crystalline cohomology relative to  $\Sigma_e = \text{Spec}(\mathbf{Z}/p^e)$  (for  $e \geq N$ ) can be recovered from crystalline cohomology relative to  $\Sigma$  via (derived) base change along  $\mathbf{Z}_p \rightarrow \mathbf{Z}/p^e$ . In general, the answer is “no,” see Example 3.11. However, under suitable flatness conditions, the answer is “yes”:

**Corollary 3.10.** *Let  $S \rightarrow X, T$  and  $\mathcal{F}$  be as in Notation 3.5 and the following discussion. Assume that for each  $e \geq N$ , the  $\mathcal{O}_{T_e}$ -module  $\mathcal{F}_{T_e}$  is flat over  $\mathbf{Z}/p^e$ . Then for each  $e \geq N$ , we have a base change isomorphism*

$$\mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L R\Gamma(S/\Sigma, \mathcal{F}) \simeq R\Gamma(S/\Sigma_e, \mathcal{F}|_{S/\Sigma_e}),$$

*where  $\mathcal{F}|_{S/\Sigma_e}$  denotes the restriction of  $\mathcal{F}$  along  $(S/\Sigma_e)_{\text{cris}} \subset (S/\Sigma)_{\text{cris}}$ .*

*Proof.* By Theorem 3.6, we have

$$R\Gamma(S/\Sigma, \mathcal{F}) \simeq R\Gamma(T, \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^1 \rightarrow \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^2 \rightarrow \cdots).$$

Since each  $\Omega_T^i$  is flat over  $\mathcal{O}_T$ , the tensor products  $\mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^i$  appearing in the de Rham complex above are automatically derived tensor products. Applying  $\mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L -$  (and observing that this operation commutes with applying  $R\Gamma(T, -)$ ) then shows that  $\mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L R\Gamma(S/\Sigma, \mathcal{F})$  is computed as the hypercohomology on  $T$  of the complex

$$K := \left( \mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L \mathcal{M} \rightarrow \mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^1 \rightarrow \mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^2 \rightarrow \cdots \right).$$

The flatness assumption on  $\mathcal{F}$  implies that  $\mathcal{M}$  is flat over  $\mathbf{Z}_p$ . Lemma 2.7 then shows that

$$\mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L \mathcal{M} \simeq \mathbf{Z}/p^e \otimes_{\mathbf{Z}} \mathcal{M} \simeq \mathcal{F}_{T_e},$$

and so

$$\mathbf{Z}/p^e \otimes_{\mathbf{Z}}^L \mathcal{M} \otimes_{\mathcal{O}_T} \Omega_T^i \simeq \mathcal{F}_{T_e} \otimes_{\mathcal{O}_{T_e}} \Omega_{T_e}^i.$$

In other words, the complex  $K$  of sheaves appearing above is identified with the de Rham complex of the  $\mathcal{O}_{T_e}$ -module  $\mathcal{F}_{T_e}$ . The claim now follows from the modulo  $p^e$  version of Theorem 3.6 (see also Remark 2.19).  $\square$

The hypotheses of Corollary 3.10 are satisfied, for example, when  $S$  is a flat local complete intersection over  $\mathbf{Z}/p^N$ , and  $\mathcal{F}$  is a crystal in *locally free* quasi-coherent  $\mathcal{O}_{S/\Sigma}$ -modules; the smooth case is discussed in [Ber74, §V.3.5]. Moreover, there are extremely simple examples illustrating the sharpness of the lci assumption:

**Example 3.11.** Let  $S = \mathbf{F}_p[x, y]/(x^2, xy, y^2)$ . In [BO83, Appendix (A.2)], Berthelot-Ogus exhibit a non-zero  $p$ -torsion class  $\tau \in H^0(S/\Sigma, \mathcal{O}_{S/\Sigma})$  by constructing a non-zero  $p$ -torsion  $\nabla$ -horizontal element of the  $p$ -adically completed divided power envelope of the natural surjection  $\mathbf{Z}_p[x, y] \rightarrow \mathbf{F}_p[x, y]/(x^2, xy, y^2)$ . Via the exact triangle

$$R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma}) \xrightarrow{p} R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma}) \rightarrow R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma}) \otimes_{\mathbf{Z}}^L \mathbf{Z}/p,$$

$\tau$  defines a non-zero class in  $H^{-1}\left(R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma}) \otimes_{\mathbf{Z}}^L \mathbf{Z}/p\right)$ . In particular,  $R\Gamma(S/\Sigma, \mathcal{O}_{S/\Sigma}) \otimes_{\mathbf{Z}}^L \mathbf{Z}/p$  has cohomology in negative degrees, so it cannot be equivalent to  $R\Gamma(S/\Sigma_1, \mathcal{O}_{S/\Sigma_1})$  (or the cohomology of any sheaf on any site).

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